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A residual bound evaluation of operator equations with Raviart-Thomas finite element

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Abstract — In this article, a residual evaluation of operator equation is considered in the framework of computer-assisted proof. Our computer-assisted approach ensures the existence and local uniqueness of weak solutions to some nonlinear partial differential equations. Based on Newton-Kantorovich theorem, our numerical method is a variant of existing methods such as [1, 2, 3, 4]. Residual evaluation for operator equation plays important role in validating numerical solutions. In order to get accurate residual evaluation, some smoothing techniques have been proposed. Main objective of this article is to obtain a sharp bound evaluation with high order Raviart-Thomas mixed finite element.

1 Introduction

Let Ω be bounded polygonal domain in $\mathbb{R}^2$ with arbitrary shape. $\mathbb{R}$ is the set of real numbers. In this article, we are concerned with Dirichlet boundary value problem of the semi-linear elliptic equation of the form:

$$\begin{cases} -\Delta u = f(\nabla u, u, x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f : H_0^1(\Omega) \to L^2(\Omega)$ is assumed to be Fréchet differentiable. For example, $f(\nabla u, u, x) = -b \cdot \nabla u - cu + c_2u^2 + c_3u^3 + g$ with $b(x) \in (L^\infty(\Omega))^2$, $c, c_2, c_3 \in L^\infty(\Omega)$ and $g \in L^2(\Omega)$ satisfies this condition. Verified computation approach will be adopted to explore the existence and local uniqueness of weak solution of (1). Namely, if an approximate solution is given by certain numerical method, we will try to validate the existence of exact solution in the neighbourhood of the approximation. In the classical analysis of variational theory, weak solution of Dirichlet boundary problem (1) is defined in variational form:

$$\text{Find } u \in H_0^1(\Omega), \quad \text{satisfying} \quad (\nabla u, \nabla v) = (f(\nabla u, u, x), v), \quad \text{for all } v \in H_0^1(\Omega). \tag{2}$$

Here,

$$(\nabla u, \nabla v) := \int_\Omega \nabla u \cdot \nabla v dx \quad \text{and} \quad (f(\nabla u, u, x), v) := \int_\Omega f(\nabla u, u, x) v dx.$$

Now we put $V = H_0^1(\Omega)$ and rewrite $f(\nabla u, u, x)$ as $f(u)$ for simple form. Let us define linear and nonlinear operators $\mathcal{A}$, $\mathcal{N} : V \to V$, $(\mathcal{A}u, v)_V := (\nabla u, \nabla v)$, $(\mathcal{N}(u), v)_V := (f(u), v)$. Furthermore, we define $\mathcal{F} : V \to V$ as $\mathcal{F}(u) := \mathcal{A}u - \mathcal{N}(u)$. The original problem (1) is equivalent to the following nonlinear operator equation:

$$\text{Find } u \in V, \quad \text{satisfying} \quad \mathcal{F}(u) = 0. \tag{3}$$

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$\mathcal{F}: V \to V$ is assumed to be Fréchet differentiable mapping. Let $\hat{u} \in V_h \subset V$ be an approximate solution to eq.(3). Fréchet derivative of $\mathcal{F}$ at $\hat{u}$ is denoted by $\mathcal{F}'[\hat{u}]: V \to V$. In order to verify the existence and local uniqueness of the exact solution in the neighborhood of $\hat{u}$, we consider to apply the Newton-Kantorovich theorem [5, 6] to eq.(3).

Theorem 1. *Assuming Fréchet derivative $\mathcal{F}'[\hat{u}]$ is nonsingular and satisfies*

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_V \leq \alpha,$$

for a certain positive α. Then, let $\overline{B}(\hat{u}, 2\alpha) := \{v \in V : \|v - \hat{u}\|_V \leq 2\alpha\}$ be a closed ball centered at $\hat{u}$ with radius 2α. Let also $D \supset \overline{B}(\hat{u}, 2\alpha)$ be an open ball in V. We assume that for a certain positive ω, it holds:

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[v] - \mathcal{F}'[w])\|_{V,V} \leq \omega\|v - w\|_V, \quad \forall v, w \in D.$$

If $\alpha\omega \leq \frac{1}{2}$ holds, then there is a solution $u \in V$ of eq.(3) satisfying

$$\|u - \hat{u}\|_V \leq \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}. \tag{4}$$

Furthermore, the solution u is unique in $\overline{B}(\hat{u}, \rho)$.

Remark 1. *To apply Newon-Kantorovich theorem, we will calculate the constants below explicitly.*

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{V,V} \leq C_1, \tag{5}$$

$$\|\mathcal{F}(\hat{u})\|_V \leq C_{2,h}, \tag{6}$$

$$\|\mathcal{F}'[v] - \mathcal{F}'[w]\|_{V,V} \leq C_3\|v - w\|_V, \quad \forall v, w \in D \subset V. \tag{7}$$

Therefore, if $C_1^2 C_{2,h} C_3 \leq 1/2$ is confirmed by verified computations, then the existence and local uniqueness of the solution are proved numerically based on Newton-Kantorovich theorem.

The main topic of this article is to evaluate the residual bound for $\mathcal{F}(\hat{u})$, *i.e.*

$$\|\mathcal{F}(\hat{u})\|_V \leq C_{2,h}. \tag{8}$$

In the following, we would like to introduce several ways to evaluate eq.(8). Suppose function $\hat{u} \in V_h$ to be an approximation of exact solution of eq.(3), where V_h is certain finite element subspace $V_h \subset V$. Our aim is to obtain *good* estimation of this residual bound. First, we introduce several evaluation methods in Section 2. Second, we show numerical results in Section 3 to demonstrate the efficiency of our proposed method. For reader's convenience, we write down the details for implementation of Raviart-Thomas element method in appendix.

2 Several ways for residual evaluation

In this section, we would like to consider the residual evaluation in the form of

$$\|\mathcal{F}(\hat{u})\|_V = \sup_{0 \neq v \in V} \frac{(\mathcal{A}\hat{u} - \mathcal{N}(\hat{u}), v)_V}{\|v\|_V} = \sup_{0 \neq v \in V} \frac{|(\nabla\hat{u}, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V}$$

in several ways. If an approximate solution satisfies $\hat{u} \in H^2(\Omega) \cap V_h$, it follows

$$\|\mathcal{F}(\hat{u})\|_V = \sup_{0 \neq v \in V} \frac{|(\nabla\hat{u}, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} = \sup_{0 \neq v \in V} \frac{|(-\Delta\hat{u}, v) - (f(\hat{u}), v)|}{\|v\|_V} \leq C_{e,2}\|\Delta\hat{u} + f(\hat{u})\|_{L^2}. \tag{9}$$

Here, $C_{e,p}$ means Sobolev's embedding constant, which satisfies $\|u\|_{L^p} \leq C_{e,p}|u|_{H^1}$, $(2 \leq p < \infty)$ for $u \in V$. We point out that the evaluation (9) does not work when V_h is taken as C^0 finite element functions, such as P_1 (piecewise linear) or P_2 (piecewise quadratic) elements. This is because $\Delta\hat{u}$ does not belong to $L^2(\Omega)$ anymore.

To weaken the condition on $\hat{u}$, we will introduce several methods that do not need the H^2-regularity of approximate solution. The first method to be introduced is fast but gives little rough bound. The second one has accurate estimation with smoothing technique. The third one is based on Raviart-Thomas mixed finite elements [9, 10, 11], which can provide better bound for residue if higher order elements are used.

2.1 Simple bounds

Let V_h be a finite element subspace of V, such that $V_h := \text{span}\{\phi_1, ..., \phi_n\}$. Let $u_h := \mathcal{P}_h u \in V_h$ be an orthogonal projection of $u \in V$, defined as $(\nabla(u - u_h), \nabla v_h) = 0$, $\forall v_h \in V_h$ In this part, we will show simple upper bound of residue. In the following, we denote v_h by the projection of v, *i.e.* $\mathcal{P}_h v$. From the classical error analysis, such as Aubin-Nitsche's trick, we have

$$\|v - v_h\|_{L^2} \le C_M \|v - v_h\|_V, \tag{10}$$

$$\|v - v_h\|_V \le \|v\|_V \quad \text{and} \quad \|v_h\|_V \le \|v\|_V. \tag{11}$$

Here C_M is a priori error constant for projection $\mathcal{P}_h$. The full discussion of this constant on arbitrary domain is shown in [12]. For $v_h \in V_h$, the residual bound of eq.(8) is given using inequalities (10) and (11)

$$\begin{aligned}
\|\mathcal{F}(\hat{u})\|_V &= \sup_{0 \ne v \in V} \frac{|(\nabla \hat{u}, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&= \sup_{0 \ne v \in V} \frac{|(\nabla \hat{u}, \nabla (v - v_h)) - (f(\hat{u}), v - v_h) + (\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v\|_V} \\
&\le \sup_{0 \ne v \in V} \frac{|(f(\hat{u}), v - v_h)|}{\|v\|_V} + \sup_{0 \ne v \in V} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v\|_V} \\
&\le C_M \|f(\hat{u})\|_{L^2} + C_r
\end{aligned} \tag{12}$$

where the quantity C_r is defined by the following procedure

$$\begin{aligned}
\sup_{0 \ne v \in V} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v\|_V} &= \sup_{\substack{0 \ne v \in V \\ 0 = v_h \in V_h}} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v\|_V} + \sup_{\substack{0 \ne v \in V \\ 0 \ne v_h \in V_h}} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v_h\|_V} \cdot \frac{\|v_h\|_V}{\|v\|_V} \\
&\le \sup_{0 \ne v_h \in V_h} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v_h\|_V} =: C_r.
\end{aligned}$$

Let ε_i be $\varepsilon_i := (\nabla \hat{u}, \nabla \phi_i) - (f(\hat{u}), \phi_i)$, $(i = 1, ..., n)$. Since $v_h \in V_h$, we can express v_h as $v_h := \sum_{i=1}^n c_i \phi_i$. Let us put $c := (c_1, ..., c_n)^t$ and $\varepsilon := (\varepsilon_1, ..., \varepsilon_n)^t$. Let further D be $n \times n$ matrix whose (i, j)-elements are given by $(\nabla \phi_i, \nabla \phi_j)$. Then, C_r follows

$$C_r = \sup_{0 \ne v_h \in V_h} \frac{|(\nabla \hat{u}, \nabla v_h) - (f(\hat{u}), v_h)|}{\|v_h\|_V} = \sup_{c \in \mathbf{R}^n} \frac{|\sum_{i=1}^n c_i \varepsilon_i|}{\sqrt{c^t D c}} \le \sup_{c \in \mathbf{R}^n} \frac{|c|_{l^2} |\varepsilon|_{l^2}}{\sqrt{c^t D c}} \le \|D^{-1}\|_2 |\varepsilon|_{l^2}. \tag{13}$$

From inequalities (12) and (13), we obtain

$$\|\mathcal{F}(\hat{u})\|_V \le C_M \|f(\hat{u})\|_{L^2} + \|D^{-1}\|_2 |\varepsilon|_{l^2}. \tag{14}$$

2.2 Accurate bounds with a smoothing technique

The simple bound (14) is a rough bound. Overestimation often causes failure in verification. Next, another method for evaluating the residual bound is introduced. This is based on the smoothing technique proposed by N. Yamamoto et. al. [13]. Here, *smoothing* means to approximate vector $\nabla \hat{u}$ by smooth function. According to [13], if P_1 (piecewise linear) elements are used for approximate solutions, the residual evaluation becomes almost the same as the rough bound in (14). On the other hand, using higher order element, this smoothing technique works very well [14]. Let $X_h \subset H^1(\Omega)$ be a finite element subspace that does not vanish on boundary of Ω. Let $p_h \in (X_h)^2$ be the vector function defined by

$$(p_h - \nabla \hat{u}, v^*) = 0, \quad \forall v^* \in (X_h)^2. \tag{15}$$

Namely it is the L^2-projection of $\nabla \hat{u} \in (L^2(\Omega))^2$ to $p_h \in (X_h)^2$. p_h makes the quantity $\|p_h - \nabla \hat{u}\|_{L^2}$ small. Further the following Green's formula holds for p_h [13]:

$$(p_h, \nabla v) + (\text{div}\, p_h, v) = 0, \quad \forall v \in V. \tag{16}$$

Therefore, using p_h and inequalities (10), (11), (13) and eq.(16), we have

$$\begin{aligned}
\|\mathcal{F}(\hat{u})\|_V &= \sup_{0\neq v\in V} \frac{|(\nabla\hat{u},\nabla v)-(f(\hat{u}),v)|}{\|v\|_V} \\
&= \sup_{0\neq v\in V} \frac{|(\nabla\hat{u},\nabla(v-v_h))-(f(\hat{u}),v-v_h)+(\nabla\hat{u},\nabla v_h)-(f(\hat{u}),v_h)|}{\|v\|_V} \\
&\le \sup_{0\neq v\in V} \frac{|(\nabla\hat{u},\nabla(v-v_h))-(f(\hat{u}),v-v_h)|}{\|v\|_V} + C_r \\
&\le \sup_{0\neq v\in V} \frac{\left|(\nabla\hat{u}-p_h,\nabla(v-v_h))+(p_h,\nabla(v-v_h))-(f(\hat{u}),v-v_h)\right|}{\|v\|_V} + C_r \\
&\le \sup_{0\neq v\in V} \frac{\|\nabla\hat{u}-p_h\|_{L^2}\|v-v_h\|_V+\|\text{div}\, p_h+f(\hat{u})\|_{L^2}\|v-v_h\|_{L^2}}{\|v\|_V} + C_r \\
&\le \|\nabla\hat{u}-p_h\|_{L^2}+C_M\|\text{div}\, p_h+f(\hat{u})\|_{L^2}+\|D^{-1}\|_2|\varepsilon|_{l^2}. \qquad (17)
\end{aligned}$$

One can use the bound (17) instead of (14). The smoothing element p_h is obtained by solving an additional linear equation (15), which takes extra computational costs. Meanwhile, for a certain *good* approximate solution, *e.g.* using P_2 (piecewise quadratic) elements, residual bound (17) becomes drastically small [14].

Remark 2. *One can consider another evaluation with $H(\text{div},\Omega)$-smoothing elements [4]. A smoothing function $q\in H(\text{div},\Omega)$ satisfying $q\approx\nabla\hat{u}$ and $\text{div}\ q+f(\hat{u})\approx 0$ yields*

$$\|\mathcal{F}(\hat{u})\|_V \le \|\nabla\hat{u}-q\|_{L^2}+C_{e,2}\|\text{div}\ q+f(\hat{u})\|_{L^2}.$$

One feature of this estimation is that it seeks the smoothing function in $q\in H(\text{div},\Omega)\supset(H^1(\Omega))^2$, which can provide better approximation of $\nabla\hat{u}$, compared with the one in eq.(15).

2.3 Raviart-Thomas mixed finite element on triangle element

Inspired by Remark 2, we are concerned with a smoothing technique using mixed finite elements as below. Here, we would like to introduce Raviart-Thomas mixed finite element [9, 10, 11]. We follow discussions in [10, 11]. Let $H(\text{div},\Omega)$ denote the space of vector functions such that

$$H(\text{div},\Omega):=\left\{\psi\in(L^2(\Omega))^2 : \text{div}\ \psi\in L^2(\Omega)\right\}.$$

Let K_h be a triangle element in triangulation of Ω. We define

$$P_k(K_h)\ :\ \text{the space of polynomials of degree less than } k \text{ on } K_h,$$

$$R_k(\partial K_h):=\{\varphi\in L^2(\partial K_h):\varphi|_{e_i}\in P_k(e_i)\},\quad \text{for any edge } e_i \text{ of } \partial K_h.$$

Functions of $R_k(\partial K_h)$ are polynomials of degree $\le k$ on each side e_i of K_h $(i=1,2,3)$. For $k\ge 0$, we define

$$RT_k(K_h)\ :=\ \left\{q\in(L^2(K_h))^2\ :\ q=\begin{pmatrix}a_k\\ b_k\end{pmatrix}+c_k\cdot\begin{pmatrix}x\\ y\end{pmatrix},\ a_k,b_k,c_k\in P_k(K_h)\right\}.$$

The dimension of $RT_k(K_h)$ is $(k+1)(k+3)$. We now introduce basic result about $RT_k(K_h)$ spaces.

Proposition 1. *Let e_i be subtense of vertex i $(=1,2,3)$ and $\vec{n}_{|e_i}=(n_1^{(i)},n_2^{(i)})^t$ be an outward unit normal vector on boundary e_i. For $q\in RT_k(K_h)$, it follows*

$$\begin{cases}\text{div}\ q\in P_k(K_h),\\ q\cdot\vec{n}_{|e_i}\in R_k(\partial K_h).\end{cases}$$

Moreover, the divergence operator is surjective from $RT_k(K_h)$ onto $P_k(K_h)$, i.e. $\text{div}(RT_k(K_h))=P_k(K_h)$.

Proposition 2. *For $k \geq 0$ and any $q \in RT_k(K_h)$, the following relations imply $q = 0$.*

$$\int_{\partial K_h} q \cdot \vec{n}\ \varphi_k ds = 0, \quad \forall \varphi_k \in R_k(\partial K_h),$$

$$\int_{K_h} q \cdot q_{k-1} dx = 0, \quad \forall q_{k-1} \in (P_{k-1}(K_h))^2.$$

The Raviart-Thomas finite element space RT_k is given by

$$RT_k \ := \ \Big\{ p_h \in (L^2(\Omega))^2 \ : \ p_h|_{K_h} = \begin{pmatrix} a_k \\ b_k \end{pmatrix} + c_k \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \ a_k, b_k, c_k \in P_k(K_h),$$
$$p_h \cdot n \text{ is continuous on the inter-element boundaries.} \Big\}$$

It is a finite dimensional subspace of $H(\mathrm{div}, \Omega)$. Further let us define $M_h := \{v \in L^2(\Omega) \ : \ v|_{K_h} \in P_k(K_h)\}$. It follows $\mathrm{div}(RT_k) = M_h$ (cf. Chapter IV.1 of [11]).

2.4 A residual bound with RT_k element

For the residual bound estimation, the smoothing technique in Subsection 2.2 works well to give accurate bounds. Some general smoothing techniques have been proposed in [2, 4, 13], etc, where smoothing functions $p_h \in (H^1(\Omega))^2$ or $H(\mathrm{div}, \Omega)$ are often used. One feature of proposal method is that we can use the basic property of Raviart-Thomas element, $\mathrm{div}(RT_k) = M_h$, for getting effective residual estimation. For given $f_h \in M_h$, this property enbables us to define a subspace of RT_k as

$$W_{f_h} = \{ \ p_h \in RT_k : \mathrm{div}\ p_h + f_h = 0\}.$$

Furthermore, we define $v_h \in M_h$ by an orthogonal projection of $v \in L^2(\Omega)$ such that $(v - v_h, w_h) = 0, \ \ \forall w_h \in M_h$. Assuming an error estimate $\|v - v_h\|_{L^2} \leq C_{M_h} \|v\|_V$ for $v_h \in M_h$ is obtained. Also we define $f_h(\hat{u}) \in M_h$ by the projection of $f(\hat{u}) \in L^2(\Omega)$. Finally, inequalities (10) and (11) give the following evaluation of the residual bound using $p_h \in W_{f_h(\hat{u})}$,

$$\begin{aligned}
\|\mathcal{F}(\hat{u})\|_V &= \sup_{0 \neq v \in V} \frac{|(\nabla \hat{u}, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&= \sup_{0 \neq v \in V} \frac{|(\nabla \hat{u} - p_h, \nabla v) + (p_h, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&\leq \sup_{0 \neq v \in V} \frac{|(\nabla \hat{u} - p_h, \nabla v)|}{\|v\|_V} + \sup_{0 \neq v \in V} \frac{|(\mathrm{div}\ p_h + f(\hat{u}), v)|}{\|v\|_V} \\
&\leq \|\nabla \hat{u} - p_h\|_{L^2} + \sup_{0 \neq v \in V} \frac{|(\mathrm{div}\ p_h + f_h(\hat{u}) + f(\hat{u}) - f_h(\hat{u}), v)|}{\|v\|_V} \\
&= \|\nabla \hat{u} - p_h\|_{L^2} + \sup_{0 \neq v \in V} \frac{|(f(\hat{u}) - f_h(\hat{u}), v - v_h)|}{\|v\|_V} \\
&\leq \|\nabla \hat{u} - p_h\|_{L^2} + C_{M_h} \|f(\hat{u}) - f_h(\hat{u})\|_{L^2}.
\end{aligned} \tag{18}$$

Remark 3. *Proposed estimation (18) holds for $k \geq 0$. If the approximate solution $\hat{u}$ is obtained from V_h, which has member function to be piecewise $(k+1)$-th polynomial. An effective choice of functional space W_{f_h} is to choose W_{f_h} is subspace of RT_k and M_h spanned by P_k elements. The rate of convergence can be expect to be $\|\nabla \hat{u} - p_h\|_{L^2} = o(h^{k+1})$ and $\|f - f_h\|_{L^2} = o(h^{k+1})$.*

3 Computational result

Now we will present numerical results to illustrate our method. All computations are carried out on Mac OS X 10.6.7, 2×2.4 GHz Quad-Core Intel Xeon (Westmere) with 64GB RAM by using MATLAB 2011a with a toolbox for verified computations, INTLAB [16]. We use Gmsh [17] (`http://geuz.org/gmsh/`) to obtain triangular mesh. Let us treat the following model problem. Here, Ω is assumed to be hexagonal domain,

$$\begin{cases} -\Delta u = u^2 + 10, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

There are two approximate solutions $\hat{u}_1, \hat{u}_2 \in V_h$ given by finite element method. These are displayed in Figure 1, 2 with the mesh size 2^{-4}. For the first approximate solution $\hat{u}_1$, verification results are shown in Table 1, 2. Here, we use P_1 (piecewise linear) and P_2 (piecewise quadratic) elements for getting $\hat{u}_1$. We adopt RT_0 space for P_1-element and RT_1 space for P_2-element.

Comparing two cases in Table 1 and Table 2, we can observe that higher order elements yield improved result .

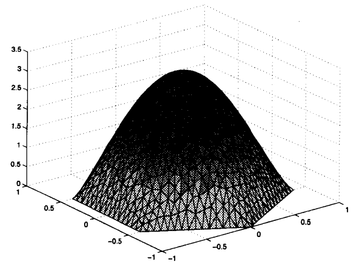

Figure 1: $\hat{u}_1$ (mesh size $\frac{1}{16}$)

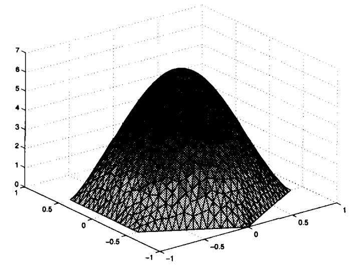

Figure 2: $\hat{u}_2$ (mesh size $\frac{1}{16}$)

Table 1: $\hat{u}_1$: P_1, $p_h \in RT_0$

$2^{-\eta}$	$\|\nabla \hat{u}_1 - p_h\|_{L^2}$	$C_{2,h}$	ρ
3	0.8535529	0.8784776	Failed
4	0.4386991	0.4448818	Failed
5	0.2285006	0.2300250	Failed
6	0.1133988	0.1137812	0.4449100

Table 2: $\hat{u}_1$: P_2, $p_h \in RT_1$

$2^{-\eta}$	$\|\nabla \hat{u}_1 - p_h\|_{L^2}$	$C_{2,h}$	ρ
3	0.1157183	0.1177798	0.5753173
4	0.0388055	0.0390525	0.1437490
5	0.0164818	0.0165182	0.0573078
6	Failed due to out of memory		

Next, we present results with respect to $\hat{u}_2$ which is from P_2 finite element space. In Table 3, comparison of each evaluation (14), (17) and (18) implies our proposed one works well. Numeric values on last column in Table 3 express upper bound of absolute error ρ using (18) residual bounds. Based on Newton-Kantorovich theorem, we prove that there is a solution in $\overline{B}(\hat{u}, \rho)$.

Table 3: Residual evaluations for $\hat{u}_2$

$2^{-\eta}$	(14)	(17)	(18)	ρ
3	8.8164705	0.6179577	0.1838180	Failed
4	4.4524279	0.3107222	0.0587483	0.2363734
5	2.1723425	0.1541978	0.0243075	0.0863220

A Notes of Raviart-Thomas elements on triangle

In this part, we would like to note representations of the lowest (RT_0) and 1st order (RT_1) Raviart-Thomas element on a triangle element K_h. Vertices of K_h are numbered as 1, 2, 3. Their coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3). Let us denote $a_i = x_j y_k - x_k y_j$, $b_i = y_j - y_k$, $c_i = x_k - x_j$ where (i, j, k) are even permutation of $(1, 2, 3)$. Here, we put subtense of each vertex as e_i with direction from j to k. See K_h in Figure 3. Then it follows

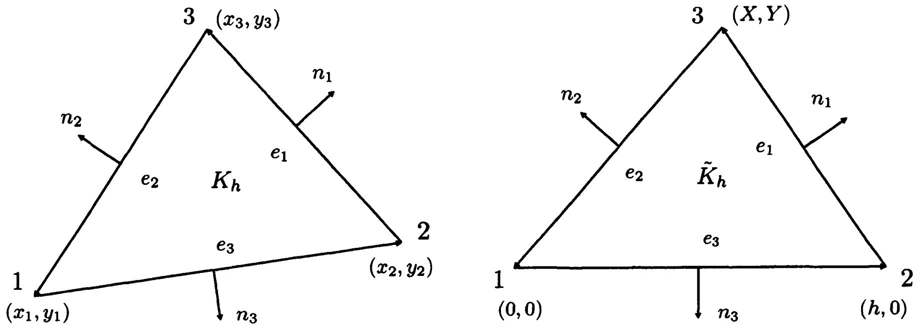

Figure 3: Triangle elements K_h and $\tilde{K}_h$

$$|e_i| = (b_i^2 + c_i^2)^{1/2}, \quad D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = b_j c_k - b_k c_j.$$

Furthermore, the unit normal vector n_i on each side is given by

$$n_i = \begin{pmatrix} n_1^{(i)} \\ n_2^{(i)} \end{pmatrix} = \frac{-\sigma}{|e_i|} \begin{pmatrix} b_i \\ c_i \end{pmatrix},$$

where $\sigma = D/|D|$ is corresponding to the direction of numbering. Namely,

$$\sigma = \begin{cases} 1, & (i, j, k : \text{ counter clockwise rotation}), \\ -1, & (i, j, k : \text{ clockwise rotation}). \end{cases}$$

For $q \in RT_k(K_h)$, degrees of freedom are given by

$$\int_{\partial K_h} q \cdot n \, \varphi_k ds, \quad \varphi_k \in R_k(\partial K_h), \quad \text{for } k \geq 0, \tag{19}$$

$$\int_{K_h} q \cdot q_{k-1} ds, \quad q_{k-1} \in (P_{k-1}(K_h))^2, \quad \text{for } k \geq 1. \tag{20}$$

A.1 RT_0 element

For $p_h \in RT_0$, the representation of RT_0 element p_h on a triangle K_h is given by

$$p_h|_{K_h} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \alpha_3 \begin{pmatrix} x \\ y \end{pmatrix}$$

Let us explain how to determine coefficients α_i. Three freedoms are given by the following form, which is equivalent to (19) in case of $k = 0$.

$$\gamma_i = |e_i| \, p_h \cdot n_i$$

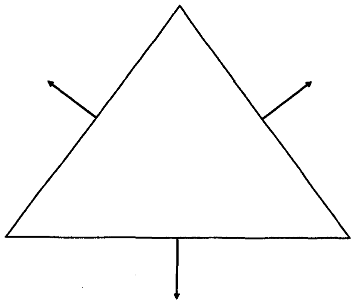

Figure 4: $RT_0(K_h)$

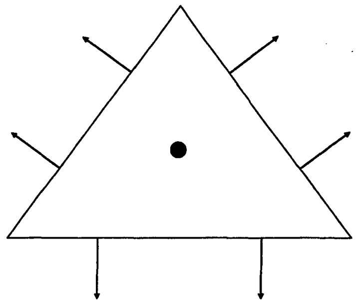

Figure 5: $RT_1(K_h)$

Notice that $p_h \cdot n_i = p_h|_{(x_j,y_j)} \cdot n_i$, we have

$$\begin{bmatrix} n_1^{(1)} & n_2^{(1)} & x_2 n_1^{(1)} + y_2 n_2^{(1)} \\ n_1^{(2)} & n_2^{(2)} & x_3 n_1^{(2)} + y_3 n_2^{(2)} \\ n_1^{(3)} & n_2^{(3)} & x_1 n_1^{(3)} + y_1 n_2^{(3)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \gamma_1/|e_1| \\ \gamma_2/|e_2| \\ \gamma_3/|e_3| \end{bmatrix} \iff \sigma \begin{bmatrix} -b_1 & -c_1 & a_1 \\ -b_2 & -c_2 & a_2 \\ -b_3 & -c_3 & a_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

Using facts for $i = 1, 2, 3$,

$$\begin{cases} \sum a_i = D, & \sum b_i = \sum c_i = 0, \\ \sum b_i x_i = D, & \sum a_i x_i = \sum c_i x_i = 0, \\ \sum c_i y_i = D, & \sum a_i y_i = \sum b_i y_i = 0, \end{cases}$$

and $\sigma D = |D|$, we have

$$\alpha_1 = -\frac{\sum \gamma_i x_i}{|D|}, \quad \alpha_2 = -\frac{\sum \gamma_i y_i}{|D|}, \quad \alpha_3 = \frac{\sum \gamma_i}{|D|}.$$

Therefore, RT_0 element on K_h can be expressed with freedoms γ_i

$$p_h|_{K_h} = \sum_{i=1}^{3} \frac{\gamma_i}{|D|} \begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix} = \sum_{i=1}^{3} \gamma_i \psi_i,$$

where ψ_i are base functions of RT_0 finite element space.

Remark 4. *The image of $RT_0(K_h)$ is given in Figure 4. Further for $q \in (L^2(\Omega))^2$, let us define linear functional, $F_i(q) = |e_i|\{q(x_j, y_j) \cdot n_i\}$ $(i = 1, 2, 3)$. It follows*

$$F_i(\psi_j) = \delta_{ij} = \begin{cases} 1, & (i = j), \\ 0, & (i \neq j), \end{cases} \qquad 1 \leq i, j \leq 3.$$

A.2 RT_1 element

Next let us consider 1st order Raviart-Thomas finite element. Degrees of freedom are denoted by $\gamma_i \in \mathbb{R}$ $(i = 1, ..., 8)$. For simplicity, we will transform triangle K_h to $\tilde{K}_h$, which has vertices $(0,0)$, $(h,0)$, (X,Y) in Figure 3.

$$h = (b_3^2 + c_3^2)^{1/2}, \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{h} \begin{pmatrix} c_3 & -b_3 \\ b_3 & c_3 \end{pmatrix} \begin{pmatrix} -c_2 \\ b_2 \end{pmatrix}, \quad D = hY,$$

$$n_1 = \frac{\sigma}{|e_1|} \begin{pmatrix} Y \\ -(X-h) \end{pmatrix}, \quad n_2 = \frac{\sigma}{|e_2|} \begin{pmatrix} -Y \\ X \end{pmatrix}, \quad n_3 = \frac{\sigma}{|e_3|} \begin{pmatrix} 0 \\ -h \end{pmatrix}.$$

In the following, we would like to explain RT_1 element on $\tilde{K}_h$. RT_1 element p_h is represented on $\tilde{K}_h$,

$$p_h|_{\tilde{K}_h} = \begin{pmatrix} \alpha_1 + \alpha_2 x + \alpha_3 y \\ \alpha_4 + \alpha_5 x + \alpha_6 y \end{pmatrix} + (\alpha_7 x + \alpha_8 y) \begin{pmatrix} x \\ y \end{pmatrix}.$$

Coefficients α_i are obtained by the following method of determination with respect to γ_i. For $i = 1, 2, 3$, degrees of freedom are given by (19) and (20),

$$\int_{e_i} p_h \cdot n_i \phi_j ds = \gamma_i, \quad \int_{e_i} p_h \cdot n_i \phi_k ds = \gamma_{i+3}, \quad \int_{\tilde{K}_h} p_h \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds = \gamma_7, \quad \int_{\tilde{K}_h} p_h \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \gamma_8,$$

where ϕ_j, ϕ_k denote piecewise linear functions on e_i, satisfying $\phi_j(x_j, y_j) = \phi_k(x_k, y_k) = 1$, $\phi_j(x_k, y_k) = \phi_k(x_j, y_j) = 0$. So that we have

$$\frac{\sigma}{6} \begin{bmatrix} 3Y & Y(X+2h) & Y^2 & -3(X-h) & -(X-h)(X+2h) & -(X-h)Y & hY(X+2h) & hY^2 \\ -3Y & -2XY & -2Y^2 & 3X & 2X^2 & 2XY & 0 & 0 \\ 0 & 0 & 0 & -3h & -h^2 & 0 & 0 & 0 \\ 3Y & Y(2X+h) & 2Y^2 & -3(X-h) & -(X-h)(2X+h) & -2(X-h)Y & hY(2X+h) & 2hY^2 \\ -3Y & -XY & -Y^2 & 3X & X^2 & XY & 0 & 0 \\ 0 & 0 & 0 & -3h & -2h^2 & 0 & 0 & 0 \\ 6 & 2(X+h) & 2Y & 0 & 0 & 0 & h^2+hX+X^2 & (2X+h)Y/2 \\ 0 & 0 & 0 & 6 & 2(X+h) & 2Y & (2X+h)Y/2 & Y^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ 2\gamma_7/D \\ 2\gamma_8/D \end{bmatrix}.$$

Solving above linear system, we have the value of each coefficients. Then, RT_1 element is described on $\tilde{K}_h$,

$$p_h|_{\tilde{K}_h} = \sum_{i=1}^{8} \gamma_i \psi_i,$$

where ψ_i are base functions as following

$$\begin{aligned}
\psi_1 &= \frac{2}{|D|} \begin{pmatrix} -2x + \frac{X}{Y}y + \frac{4}{h}(x^2 - \frac{X}{Y}xy) \\ -y + \frac{4}{h}(xy - \frac{X}{Y}y^2) \end{pmatrix}, \\
\psi_2 &= \frac{2}{|D|} \begin{pmatrix} h - x - (\frac{X+3h}{Y})y + \frac{4}{Y}xy \\ -2y + \frac{4}{Y}y^2 \end{pmatrix}, \\
\psi_3 &= \frac{2}{|D|} \begin{pmatrix} -2X + 3(\frac{X+3h}{h})x - \frac{3X}{D}(X-h)y + \frac{4}{h}(-x^2 + (\frac{X-h}{Y})xy) \\ -2Y + \frac{3Y}{h}x - \frac{3}{h}(X-2h)y + \frac{4}{h}(-xy + (\frac{X-h}{Y})y^2) \end{pmatrix}, \\
\psi_4 &= \frac{2}{|D|} \begin{pmatrix} -x - \frac{X}{Y}y + \frac{4}{Y}xy \\ -2y + \frac{4}{Y}y^2 \end{pmatrix}, \\
\psi_5 &= \frac{2}{|D|} \begin{pmatrix} -2h + 6x - 3(\frac{X-h}{Y})y + \frac{4}{h}(-x^2 + (\frac{X-h}{Y})xy) \\ 3y + \frac{4}{h}(-xy + (\frac{X-h}{Y})y^2) \end{pmatrix}, \\
\psi_6 &= \frac{2}{|D|} \begin{pmatrix} X - (\frac{3X+2h}{h})x + \frac{X}{D}(3X+h)y + \frac{4}{h}(x^2 - \frac{X}{Y}xy) \\ Y - \frac{3Y}{h}x + (\frac{3X-h}{h})y + \frac{4}{h}(xy - \frac{X}{Y}y^2) \end{pmatrix}, \\
\psi_7 &= \frac{8}{h|D|} \begin{pmatrix} 2x - \frac{X}{Y}y - \frac{2}{h}x^2 + (\frac{2X-h}{D})xy \\ y - \frac{2}{h}xy + (\frac{2X-h}{D})y^2 \end{pmatrix}, \\
\psi_8 &= \frac{8}{D|D|} \begin{pmatrix} -(2X-h)x + \frac{X}{Y}(X+h)y + (\frac{2X-h}{h})x^2 - 2(\frac{X^2-Xh+h^2}{D})xy \\ -(X-2h)y + (\frac{2X-h}{h})xy - 2(\frac{X^2-Xh+h^2}{D})y^2 \end{pmatrix}.
\end{aligned}$$

Remark 5. *See Figure 5 for degrees of freedom to $RT_1(\tilde{K}_h)$. A linear functional is defined by $F_i(q)$, $(i = 1, ..., 8)$ for $q \in (L^2(\Omega))^2$, such that*

$$F_l(q) = \int_{e_l} q \cdot n_l \phi_m ds, \quad F_{l+3}(q) = \int_{e_l} q \cdot n_l \phi_n ds, \quad F_7(q) = \int_{\tilde{K}_h} q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx, \quad F_8(q) = \int_{\tilde{K}_h} q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx$$

where (l, m, n) are even permutation of $(1, 2, 3)$. Then, we have $F_i(\psi_j) = \delta_{ij}$, $(1 \leq i, j \leq 8)$.

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